Enrollment No: ____________________ Exam Seat No: ____________________

C.U.SHAH UNIVERSITY

Summer Examination-2017

Subject Name : Linear Algebra

Subject Code : 5SC01LIA1 **Branch: M.Sc(Mathematics)**

Semester : 1 **Date : 20/03/2017** **Time : 10:30 To 01:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

a. Define: Idempotent linear transformation

b. Determine the dimension and basis for the following subspace of R^3:
The plane $3x-2y+5z=0$

c. Determine whether the given vectors $u=(-4,6,-10,1)$ and $v=(2,1,-2,9)$ are orthogonal with respect to the Euclidean inner product.

d. Find the coordinate vector of $p=2-x+x^2$ relative to the basis $\mathrm{S}=\{p_1,p_2,p_3\}$ where $p_1=1+x,\ p_2=1+x^2,\ p_3=x+x^2$

e. Check whether the set of vectors of the form $(a,2,0)$ is a subspace of R^3 or not.

f. Find $d(u,v)$ if $u=(-1,2)$, $v=(2,\ 5)$ and weighted Euclidean inner product is $\langle u,v\rangle=3u_1v_1+2u_2v_2$ where $u=(u_1,u_2)$ and $v=(v_1,v_2)$.

g. Determine algebraic multiplicity of each eigen value of the matrix $\begin{bmatrix}4&5&0\\0&4&5\\0&0&4\end{bmatrix}$.

Q-2 **Attempt all questions** **(14)**

a. Find eigen value and eigen vectors for the linear transformation $T:R^3\to R^3$ defined as $T(x,y,z)=(-2y-2z,-2x-3y-2z,3x+6y+5z);\ \forall(x,y,z)\in R^3$ by considering the standard basis of R^3. **(5)**

b. Let n be any positive integer. Consider the set $V=R^n$ of all vectors of the form $u=(u_1,u_2,..........,u_n)$ together with standard vector addition $u+v=(u_1+v_1,u_2+v_2,..........,u_n+v_n)$ and scalar multiplication $cu=(cu_1,cu_2,.........,cu_n)$ where $u,v\in R^n;\ u_1,u_2,.........,u_n\in R;$ **(5)**

$v_1, v_2, ..., v_n \in R$, $c \in R$. Show that V is a vector space.

c. Express the polynomial $p = -9 - 7x - 15x^2$ as a linear combination of $p_1 = 2 + x + 4x^2$, $p_2 = 1 - x + 3x^2$ and $p_3 = 3 + 2x + 5x^2$ **(4)**

OR

Q-2 **Attempt all questions** **(14)**

a. State and prove Riesz Representation Theorem. **(5)**

b. Show that if V is a finite dimensional vector space then any two basis of V have the same number of elements. **(5)**

c. Find $N(T)$, $n(T)$, $R(T)$, $r(T)$ for the following linear transformation T. **(4)**

$T : R^2 \to R^3;\ T(x, y) = (x, x + y, y);\ \forall (x, y) \in R^2$

Q-3 **Attempt all questions** **(14)**

a. Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis vectors $u_1 = (1,0,0)$, $u_2 = (3,7,-2)$, $u_3 = (0,4,1)$ into an orthonormal basis. **(5)**

b. State and prove Rank-Nullity theorem. **(5)**

c. Check whether the set of vectors (1, 1, 2, 1), (1, 0 , 0, 2),(4, 6, 8, 6) and (0, 3, 2, 1) are linearly dependent or not. **(4)**

OR

Q-3 **Attempt all questions** **(14)**

a. Check whether the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ is diagonalizable or not. **(5)**

b. If $T : U \to V$ and $S : V \to W$ be any non-singular linear transformation, then prove that $(S \circ T) : U \to W$ is also non-singular and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$. **(5)**

c. State and prove Cayley – Hamilton theorem. **(4)**

SECTION – II

Q-4 **Attempt the Following questions (1 Mark *7=7)** **(07)**

a. Show that if A is invertible then for all B, $\det(ABA^{-1}) = \det B$.

b. Find the minimal polynomial for the matrix $A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}_{7\times7}$

c. What is Index of nilpotence of $T \in A(V)$?

d. Define: Orthogonal matrix

e. Give an example of Skew Hermitian matrix.

f. Express the following quadratic forms in matrix notation.

$x_1^2 + 2x_2^2 + 3x_3^2 + x_4^2 - 2x_1x_2 + 4x_1x_3 - 2x_1x_4 + 4x_2x_3 - 6x_2x_4 - 8x_3x_4$

g. Write down the quadratic form corresponding to the following matrix.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

Q-5 **Attempt all questions** **(14)**

a. Show that for $A, B \in F_n$, $\det(AB) = (\det A)(\det B)$. **(5)**

b. Define Jordan Canonical form. Determine all possible Jordan canonical forms for a linear operator $T : V \to V$ whose characteristic polynomial is $\Delta(t) = (t-7)^4$. **(5)**

c. Show that interchanging two rows of A changes the sign of its determinant. **(4)**

OR

Q-5 **Attempt all questions** **(14)**

a. Reduce the quadratic form $Q = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$ to canonical form by congruent transformation. **(5)**

b. Show that if V is a n dimensional over F and if $T \in A(V)$ has all its characteristic roots in F, then T satisfies a polynomial of degree n over F. **(5)**

c. Describe the conic whose equation is $9x^2 + 4y^2 - 36x - 24y + 36 = 0$. Give it's equation in the translated coordinate system. **(4)**

Q-6 **Attempt all questions** **(14)**

a. Prove that two nilpotent linear transformations are similar if and only if they have the same invariants. **(5)**

b. If $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices then prove that **(5)**

(i) The trace is a linear operator.

(ii) The trace is a commutative operator.

(iii) $Tr(A^T) = Tr(A)$

c. The Characteristic and minimal polynomials of an operator T are respectively, $\Delta(t) = (t-5)^4 (t-6)^3$ and $m(t) = (t-5)^2 (t-6)^2$. Find all possible Jordan canonical forms with above conditions. **(4)**

OR

Q-6 **Attempt all Questions** **(14)**

a. If $T \in A(V)$ has all its characteristic roots in F, then prove that there is a basis of V in which the matrix of T is triangular. **(5)**

b. Describe the conic whose equation is $9x^2 + 6xy + y^2 - 4 = 0$. Give it's equation in the rotated coordinate system. **(5)**

c. Show that the characteristic roots of a Hermitian matrix are all real. **(4)**